Eur. Phys. J. C 50, 683–689 (2007)
DOI 10.1140/epjc/s10052-007-0224-8

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

The geometrical form for the string space-time action[a]

D.S. Popović[b], B. Sazdović[c]

Institute of Physics, 11001 Belgrade, P.O. Box 57, Serbia

Received: 26 July 2006 / Revised version: 9 October 2006 /
Published online: 16 February 2007 –

Abstract. In the present article, we derive the space-time action of the bosonic string in terms of geometrical quantities. First, we study the space-time geometry felt by a probe bosonic string moving in antisymmetric and dilaton background fields. We show that the presence of the antisymmetric field leads to space-time torsion, and the presence of the dilaton field leads to space-time non-metricity. Using these results we obtain the integration measure for space-time with stringy non-metricity, requiring its preservation under parallel transport. We derive the Lagrangian depending on stringy curvature, torsion and non-metricity.

1 Introduction

General relativity is described in terms of a torsion free and metric compatible connection. There are many generalizations of this theory that include a non-trivial contribution of torsion and non-metricity [1, 2].

We are interested in the theory of gravity obtained from string theory, describing the massless states of the closed bosonic string. Besides the metric tensor $G_{\mu\nu}$, it contains the antisymmetric tensor $B_{\mu\nu}$ and the dilaton field Φ. The space-time field equations of this theory can be derived from the requirement of Weyl invariance of the quantum world-sheet theory, as a condition of consistent string theory [3–7]. It is a non-trivial fact that these field equations can be obtained from a single space-time action. Consequently, the quantum conformal invariance of the world-sheet leads to generalized space-time Einstein equations and the corresponding action. The question is whether there exists a geometrical interpretation of this action. In our interpretation, it means the existence of a generalized connection, so that the above action can be written in terms of corresponding generalized curvature, torsion and non-metricity.

There have been many attempts to achieve this goal of expressing this action in terms of geometrical quantities. In [8–10] there is a restriction on the number of space-time dimensions ($D = 2$ and $D = 4$), in order to use the Hodge dual map. Some articles [8, 11, 12] investigate a Riemann–Cartan metric compatible space-time, while in [9] the space-time is torsion free but with a connection non-compatible with the metric. In the article [10], one considers space-time with both torsion and non-metricity non-trivial. The authors of this reference assume a certain form for the torsion and non-metricity, and they restrict their considerations to $D = 4$ space-time dimensions. The torsion is usually connected with the field strength of an antisymmetric field [13–15], while in some papers [11, 12] the trace of the torsion is related to the gradient of the dilaton field.

In this article we first derive the form of the connection from the world-sheet equations of motion. It corresponds to a new covariant derivative, which makes it easier to perform calculations, including those in a quantization procedure. We find that the string sees the space-time not as a Riemann one, but as some particular form of affine space-time, which besides curvature also depends on torsion and non-metricity. The features of this geometry define effective general relativity in target space.

In Sect. 2, we formulate the theory and briefly repeat some results of [16].

Starting with the known rules of space-time parallel transport, in Sect. 3 we introduce the torsion and non-metricity. We decompose the arbitrary connection in terms of the Christoffel one, contortion and non-metricity. With the help of the equations of motion, we find a particular form of stringy torsion and stringy non-metricity [16]. To the space-time felt by the probe string we will refer as stringy space-time.

In Sect. 4, we derive the form of the space-time action. We obtain the integration measure for spaces with non-metricity from the requirements that the measure is preserved under parallel transport, and that it enables integration by parts. Our integration measure is a volume-form compatible with the affine connection [17]. In a particular application, to improve the standard measure, [12] uses the torsion, while we use the non-metricity. We construct the Lagrangian linear in the stringy invariants of scalar curvature, square of the torsion and square of the non-metricity. We discuss the relation of the space-time

[a] Work supported in part by the Serbian Ministry of Science and Environmental Protection, under contract No. 141036.

[b] e-mail: popovic@phy.bg.ac.yu

[c] e-mail: sazdovic@phy.bg.ac.yu

action of the present paper with the space-time action of [3–7].

Appendix A is devoted to the world-sheet geometry.

2 Canonical derivation of the field equations

The closed bosonic string, propagating in an arbitrary background, is described by the sigma model (see [3–7] and [18, 19]):

$$S=\kappa\int_\Sigma \mathrm{d}^2\xi\sqrt{-g}\left\{\left[\frac{1}{2}g^{\alpha\beta}G_{\mu\nu}(x)+\frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}}B_{\mu\nu}(x)\right]\times\partial_\alpha x^\mu\partial_\beta x^\nu+\Phi(x)R^{(2)}\right\}, \quad (1)$$

with x^μ-dependent background fields: the metric $G_{\mu\nu}$, the antisymmetric tensor field $B_{\mu\nu}=-B_{\nu\mu}$ and the dilaton field Φ. Here, $g_{\alpha\beta}$ is the intrinsic world-sheet metric and $R^{(2)}$ is the corresponding scalar curvature. Let $x^\mu(\xi)$ ($\mu=0,1,...,D-1$) be the coordinates of the D dimensional space-time M_D and ξ^α ($\xi^0=\tau,\xi^1=\sigma$) the coordinates of the two dimensional world-sheet Σ spanned by the string. We will denote the corresponding derivatives as $\partial_\mu\equiv\frac{\partial}{\partial x^\mu}$ and $\partial_\alpha\equiv\frac{\partial}{\partial\xi^\alpha}$.

Let us briefly review the canonical analysis and derivation of the field equations obtained in [16]. Restricting our considerations to the condition $a^2\equiv G^{\mu\nu}a_\mu a_\nu\neq 0$ ($a_\mu=\partial_\mu\Phi$), we define the currents

$$J_{\pm\mu}=P^T{}_\mu{}^\nu j_{\pm\nu}+\frac{a_\mu}{2a^2}i^\Phi_\pm=j_{\pm\mu}-\frac{a_\mu}{a^2}j\,, \quad (2)$$

$$i^F_\pm=\frac{a^\mu}{a^2}j_{\pm\mu}-\frac{1}{2a^2}i^\Phi_\pm\pm 2\kappa F'\,,\quad i^\Phi_\pm=\pi_F\pm 2\kappa\Phi'\,, \quad (3)$$

where

$$j_{\pm\mu}=\pi_\mu+2\kappa\Pi_{\pm\mu\nu}x^{\nu\prime}\,,\quad \Pi_{\pm\mu\nu}\equiv B_{\mu\nu}\pm\frac{1}{2}G_{\mu\nu} \quad (4)$$

and

$$j=a^\mu j_{\pm\mu}-\frac{1}{2}i^\Phi_\pm=a^2\left(i^F_\pm\mp 2\kappa F'\right). \quad (5)$$

Here π_μ and π_F are the canonically conjugate momenta to the variables x^μ and F.

Up to the boundary term, the canonical Hamiltonian density has the standard form

$$\mathcal{H}_\mathrm{c}=h^-T_-+h^+T_+\,, \quad (6)$$

with the energy–momentum tensor components

$$T_\pm=\mp\frac{1}{4\kappa}\left(G^{\mu\nu}J_{\pm\mu}J_{\pm\nu}+i^F_\pm i^\Phi_\pm\right)+\frac{1}{2}i^{\Phi\prime}_\pm$$
$$=\mp\frac{1}{4\kappa}\left(G^{\mu\nu}j_{\pm\mu}j_{\pm\nu}-\frac{j^2}{a^2}\right)+\frac{1}{2}\left(i^{\Phi\prime}_\pm-F'i^\Phi_\pm\right). \quad (7)$$

In spite of their complicated expressions, the same chirality energy–momentum tensor components satisfy two independent copies of Virasoro algebras,

$$\{T_\pm(\sigma),T_\pm(\bar\sigma)\}=-[T_\pm(\sigma)+T_\pm(\bar\sigma)]\delta'(\sigma-\bar\sigma)\,, \quad (8)$$

while the opposite chirality components commute: $\{T_\pm,T_\mp\}=0$.

2.1 Equations of motion

In [16], using the canonical approach, we derived the following equations of motion:

$$[J^\mu]\equiv\nabla_\mp\partial_\pm x^\mu+{}^\star\Gamma^\mu_{\mp\rho\sigma}\partial_\pm x^\rho\partial_\mp x^\sigma=0\,, \quad (9)$$

$$[h^\pm]\equiv G_{\mu\nu}\partial_\pm x^\mu\partial_\pm x^\nu-2\nabla_\pm\partial_\pm\Phi=0\,, \quad (10)$$

$$[i^F]\equiv R^{(2)}+\frac{2}{a^2}(D_{\mp\mu}a_\nu)\partial_\pm x^\nu\partial_\mp x^\mu=0\,, \quad (11)$$

where the variables in the parentheses denote the currents corresponding to this equation. The expression

$${}^\star\Gamma^\rho_{\pm\nu\mu}=\Gamma^\rho_{\pm\nu\mu}+\frac{a^\rho}{a^2}D_{\pm\mu}a_\nu$$
$$=\Gamma^\rho_{\nu\mu}\pm P^{T\rho}{}_\sigma B^\sigma_{\nu\mu}+\frac{a^\rho}{a^2}D_\mu a_\nu\,, \quad (12)$$

which appears in the equation for $[J^\mu]$, is a generalized connection, the full geometrical interpretation of which we are going to investigate. Under general space-time coordinate transformations, the expression ${}^\star\Gamma^\rho_{\pm\nu\mu}$ transforms as a connection.

The covariant derivatives with respect to the Christoffel connection $\Gamma^\rho_{\nu\mu}$ and to the connection $\Gamma^\rho_{\pm\nu\mu}=\Gamma^\rho_{\nu\mu}\pm B^\rho_{\nu\mu}$ we respectively denote as D_μ and $D_{\pm\mu}$, while

$$B_{\mu\nu\rho}=\partial_\mu B_{\nu\rho}+\partial_\nu B_{\rho\mu}+\partial_\rho B_{\mu\nu}$$
$$=D_\mu B_{\nu\rho}+D_\nu B_{\rho\mu}+D_\rho B_{\mu\nu} \quad (13)$$

is the field strength of the antisymmetric tensor. The projection operator which appears in (12),

$$P^T{}_{\mu\nu}=G_{\mu\nu}-\frac{a_\mu a_\nu}{a^2}\equiv G^{D-1}_{\mu\nu}\,, \quad (14)$$

is the induced metric on the $D-1$ dimensional submanifold defined by the condition $\Phi(x)=\mathrm{const}$.

In (9) and (11) we omit the $\pm$ indices of the currents, because $[J^\mu_+]=[J^\mu_-]$ and $[i^F_+]=[i^F_-]$ as a consequence of the symmetry relations ${}^\star\Gamma^\mu_{\mp\rho\sigma}={}^\star\Gamma^\mu_{\pm\sigma\rho}$ and $D_{\mp\mu}a_\nu=D_{\pm\nu}a_\mu$.

3 The geometry of space-time seen by the probe string

In this section we introduce the affine linear connection, the torsion and the non-metricity [1, 2]. With the help of the string field equations we derive expressions for the stringy connection, torsion and non-metricity, felt by the probe string.

3.1 Geometry of space-time with torsion and non-metricity

In curved spaces, the operations on tensors are covariant only if they are realized in the same point. In order to compare vectors from different points, we need the rule for parallel transport. The parallel transport of the vector $V^\mu(x)$ from the point x to the point $x+\mathrm{d}x$ produces the vector ${}^\circ V^\mu_\| = V^\mu + {}^\circ\delta V^\mu$, where

$$ {}^\circ\delta V^\mu = -{}^\circ\Gamma^\mu_{\rho\sigma} V^\rho \, \mathrm{d}x^\sigma \, . \tag{15} $$

The variable ${}^\circ\Gamma^\mu_{\rho\sigma}$ is the *affine linear connection*. The covariant derivative is defined in the standard form

$$ \begin{aligned} {}^\circ D V^\mu &= V^\mu(x+\mathrm{d}x) - {}^\circ V^\mu_\| = \mathrm{d}V^\mu - {}^\circ\delta V^\mu \\ &= \left(\partial_\nu V^\mu + {}^\circ\Gamma^\mu_{\rho\nu} V^\rho\right) \mathrm{d}x^\nu \equiv {}^\circ D_\nu V^\mu \, \mathrm{d}x^\nu \, . \end{aligned} \tag{16} $$

The antisymmetric part of the affine connection is the *torsion*:

$$ {}^\circ T^\rho_{\mu\nu} = {}^\circ\Gamma^\rho_{\mu\nu} - {}^\circ\Gamma^\rho_{\nu\mu} \, . \tag{17} $$

It has a simple geometrical interpretation, because it measures the non-closure of the curved "parallelogram".

The *metric tensor* $G_{\mu\nu}$ is an independent variable that enables calculation of the scalar product $VU = G_{\mu\nu} V^\mu U^\nu$, allowing one to measure lengths and angles.

We already learnt that the covariant derivative is responsible for the comparison of vectors at different points. What variable is responsible for comparison of the lengths of these vectors? The squares of the lengths of the vectors, $V^\mu(x)$ and its parallel transport to the point $x+\mathrm{d}x$, ${}^\circ V^\mu_\|$, are defined respectively as $V^2(x) = G_{\mu\nu}(x) V^\mu(x) V^\nu(x)$ and ${}^\circ V^2_\|(x+\mathrm{d}x) = G_{\mu\nu}(x+\mathrm{d}x) {}^\circ V^\mu_\| \, {}^\circ V^\nu_\|$. If we remember the invariance of the scalar product under parallel transport, then the difference of the squares of the vectors is

$$ \begin{aligned} {}^\circ\delta V^2 &= {}^\circ V^2_\|(x+\mathrm{d}x) - V^2(x) \\ &= \left[G_{\mu\nu}(x+\mathrm{d}x) - G_{\mu\nu}(x) - {}^\circ\delta G_{\mu\nu}(x)\right] {}^\circ V^\mu_\| \, {}^\circ V^\nu_\| \, . \end{aligned} \tag{18} $$

Up to higher-order terms we have

$$ \begin{aligned} {}^\circ\delta V^2 &= \left[\mathrm{d}G_{\mu\nu}(x) - {}^\circ\delta G_{\mu\nu}(x)\right] V^\mu V^\nu \\ &= {}^\circ D G_{\mu\nu} V^\mu V^\nu \equiv -\mathrm{d}x^\rho \, {}^\circ Q_{\rho\mu\nu} V^\mu V^\nu \, , \end{aligned} \tag{19} $$

where we introduced the *non-metricity* as a covariant derivative of the metric tensor:

$$ {}^\circ Q_{\mu\rho\sigma} = -{}^\circ D_\mu G_{\rho\sigma} \, . \tag{20} $$

Besides the length, the non-metricity also changes the angle between the vectors V^μ_1 and V^μ_2, according to the relation

$$ \begin{aligned} {}^\circ\delta \cos(\angle(V_1, V_2)) &= \frac{-1}{2\sqrt{V_1^2 V_2^2}} \\ \times &\left[2 V_1^\rho V_2^\sigma - \left(\frac{V_1^\rho V_1^\sigma}{V_1^2} + \frac{V_2^\rho V_2^\sigma}{V_2^2}\right)(V_1 V_2)\right] {}^\circ Q_{\mu\rho\sigma} \, \mathrm{d}x^\mu \, . \end{aligned} \tag{21} $$

Note that we performed parallel transport of the vectors, but not of the metric tensor. This means that for the length calculation in the point $x+\mathrm{d}x$ we used the metric tensor $G_{\mu\nu}(x+\mathrm{d}x)$, which lives in this point, and not the tensor $G_{\mu\nu} + {}^\circ\delta G_{\mu\nu}$ obtained after parallel transport from the point x. The requirement for the equality of these two tensors is known in the literature as the metric postulate. In fact, it is just compatibility between the metric and the connection, such that the metric after parallel transport is equal to the local metric. Here we will not accept this requirement, because the difference of these two tensors is the origin of the non-metricity. So the non-metricity measures the deformation of lengths and angles during parallel transport.

We also define the Weyl vector by

$$ {}^\circ q_\mu = \frac{1}{D} G^{\rho\sigma} \, {}^\circ Q_{\mu\rho\sigma} \, , \tag{22} $$

where D is the number of space-time dimensions. When the traceless part of the non-metricity vanishes,

$$ {}^\circ \tilde{Q}_{\mu\rho\sigma} \equiv {}^\circ Q_{\mu\rho\sigma} - G_{\rho\sigma} \, {}^\circ q_\mu = 0 \, , \tag{23} $$

the parallel transport preserves the angles but not the lengths. Such a geometry is known as a Weyl geometry.

Following [1, 2], we can decompose the connection ${}^\circ\Gamma^\mu_{\nu\rho}$ in terms of the Christoffel connection, contortion and non-metricity. If we introduce the Schouten braces according to the relation

$$ \{\mu\rho\sigma\} = \sigma\mu\rho + \rho\sigma\mu - \mu\rho\sigma \, , \tag{24} $$

then the Christoffel connection can be expressed as $\Gamma_{\mu,\rho\sigma} = \frac{1}{2}\partial_{\{\mu} G_{\rho\sigma\}}$. The contortion ${}^\circ K_{\mu\rho\sigma}$ is defined in terms of the torsion

$$ \begin{aligned} {}^\circ K_{\mu\rho\sigma} &= \frac{1}{2} {}^\circ T_{\{\sigma\mu\rho\}} \\ &= \frac{1}{2}\left({}^\circ T_{\rho\sigma\mu} + {}^\circ T_{\mu\rho\sigma} - {}^\circ T_{\sigma\mu\rho}\right) . \end{aligned} \tag{25} $$

The Schouten braces of the non-metricity can be solved in terms of the connection, producing

$$ {}^\circ\Gamma_{\mu,\rho\sigma} = \Gamma_{\mu,\rho\sigma} + {}^\circ K_{\mu\rho\sigma} + \frac{1}{2} {}^\circ Q_{\{\mu\rho\sigma\}} \, . \tag{26} $$

The first term is the Christoffel connection, which depends on the metric, but which does not transform as a tensor. The second one is the contortion (25), and the third one contains the Schouten braces of the non-metricity (20). The last two terms transform as tensors.

3.2 Stringy torsion and non-metricity

The manifold M_D, together with the affine connection ${}^\circ\Gamma^\mu_{\nu\rho}$ and the metric $G_{\mu\nu}$, define the affine space-time $A_D \equiv (M_D, {}^\circ\Gamma, G)$. We will refer to the connection (12) as the *stringy connection*, and we will refer to the corresponding space-time $S_D \equiv (M_D, {}^\star\Gamma_\pm, G)$, observed by the string propagating in the background $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, as the *stringy space-time*.

The antisymmetric part of the stringy connection is the *stringy torsion*:

$$^\star T_{\pm}{}^{\rho}_{\mu\nu} = {}^\star\Gamma_{\pm}{}^{\rho}_{\mu\nu} - {}^\star\Gamma_{\pm}{}^{\rho}_{\nu\mu} = \pm 2 P^{T\rho}{}_{\sigma} B^{\sigma}_{\mu\nu}\,. \tag{27}$$

It is the transverse projection of the field strength of the antisymmetric tensor field $B_{\mu\nu}$. The form of (13) suggests that $B_{\mu\nu}$ is a torsion potential [20, 21].

The presence of the dilaton field Φ leads to breaking of the space-time metric postulate. The non-compatibility of the metric $G_{\mu\nu}$ with the stringy connection $^\star\Gamma^{\mu}_{\pm\nu\rho}$ is measured by the *stringy non-metricity*,

$$^\star Q_{\pm\mu\rho\sigma} \equiv -{}^\star D_{\pm\mu} G_{\rho\sigma} = \frac{1}{a^2} D_{\pm\mu}(a_\rho a_\sigma)\,. \tag{28}$$

Consequently, during stringy parallel transport, the length and angle deformations depend on the vector field a_μ.

The stringy Weyl vector

$$^\star q_\mu = \frac{1}{D} G^{\rho\sigma}\,{}^\star Q_{\pm\mu\rho\sigma} = \frac{-4}{D}\partial_\mu \varphi \tag{29}$$

is the gradient of the new scalar field φ, defined by the expression

$$\varphi = -\frac{1}{4}\ln a^2 = -\frac{1}{4}\ln(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi)\,. \tag{30}$$

The stringy angle preservation relation

$$^\star\mathcal{Q}_{\pm\mu\rho\sigma} = {}^\star Q_{\pm\mu\rho\sigma} - G_{\rho\sigma}\,{}^\star q_\mu = 0 \tag{31}$$

is a condition on the dilaton field Φ. Generally, in stringy geometry both the lengths and the angles could be changed under parallel transport.

Using the relation

$$^\star K_{\pm\mu\rho\sigma} + \frac{1}{2}{}^\star Q_{\pm\{\mu\rho\sigma\}} = \pm\frac{1}{2}{}^\star T_{\mu\rho\sigma} + \frac{1}{2}{}^\star Q_{\{\mu\rho\sigma\}}\,, \tag{32}$$

instead of (26) we can write

$$^\star\Gamma_{\pm\mu,\rho\sigma} = \Gamma_{\mu,\rho\sigma} \pm \frac{1}{2}{}^\star T_{\mu\rho\sigma} + \frac{1}{2}{}^\star Q_{\{\mu\rho\sigma\}}\,, \tag{33}$$

where the quantities $^\star T_{\mu\rho\sigma} = 2P^{T\nu}{}_{\mu} B_{\nu\rho\sigma}$ and $^\star Q_{\mu\rho\sigma} = -{}^\star D_\mu G_{\rho\sigma} = \frac{1}{a^2} D_\mu(a_\rho a_\sigma)$ do not depend on the $\pm$ indices. In fact, the last term is $^\star Q_{\{\mu\rho\sigma\}} = 2\frac{a_\mu}{a^2} D_\rho a_\sigma$, so that we can recognize the start expression (12).

4 The space-time action

The space-time field equations for background fields, derived as a quantum consistency condition of string theory [3–7], has the form

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2D_\mu a_\nu = 0\,, \tag{34}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2a_\rho B^\rho{}_{\mu\nu} = 0\,, \tag{35}$$

$$\beta^\Phi \equiv 4\pi\kappa\frac{D-26}{3} - R + \frac{1}{12} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - 4D_\mu a^\mu + 4a^2 = 0\,, \tag{36}$$

so that the world-sheet theory is Weyl invariant. Here $R_{\mu\nu}$, R and D_μ are the space-time Ricci tensor, scalar curvature and covariant derivative, respectively, while $B_{\mu\rho\sigma}$ is the field strength of the field $B_{\mu\nu}$ and $a_\mu = \partial_\mu\Phi$.

These field equations can be derived from a single space-time action,

$$S = \int \mathrm{d}x\sqrt{-G}\mathrm{e}^{-2\Phi}\left[R - \frac{1}{12}B^2 + 4(\partial\Phi)^2\right]\,, \tag{37}$$

where $B^2 = B_{\mu\nu\rho}B^{\mu\nu\rho}$ and $(\partial\Phi)^2 = G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$.

The action is defined up to a total derivative. So it depends on some constant parameter ζ:

$$S_\zeta = S + \zeta\int \mathrm{d}x\partial_\mu\left(\sqrt{-G}G^{\mu\nu}\partial_\nu \mathrm{e}^{-2\Phi}\right) \tag{38}$$

and can be rewritten in the form

$$S_\zeta = \int \mathrm{d}x \sqrt{-G}\mathrm{e}^{-2\Phi}\left[R - \frac{1}{12}B^2 + 4(1+\zeta)(\partial\Phi)^2 - 2\zeta D^2\Phi\right]\,, \tag{39}$$

where $D^2\Phi = G^{\mu\nu}D_\mu\partial_\nu\Phi$. For simplicity, in order to exclude the third term, we adopt $\zeta = -1$ and obtain

$$S_{\zeta=-1} = \int \mathrm{d}x\sqrt{-G}\mathrm{e}^{-2\Phi}\left[R - \frac{1}{12}B^2 + 2D^2\Phi\right] \equiv \int \mathrm{d}x\sqrt{-G}\mathrm{e}^{-2\Phi}\mathcal{L}\,. \tag{40}$$

Using the stringy geometry introduced in the previous section, we are going to reproduce the above space-time action. Generally, it has the form

$$^\star S = \int \mathrm{d}^D x\; {}^\star\Omega\; {}^\star\mathcal{L}\,, \tag{41}$$

where $^\star\Omega$ is a measure factor, and $^\star\mathcal{L}$ is a Lagrangian which depends on the space-time field strengths.

4.1 The space-time measure

We define the invariant measure, requiring that the following holds.

1. It is invariant under space-time general coordinate transformations.
2. It is preserved under parallel transport, which is equivalent to the condition $^\star D_{\pm\mu}{}^\star\Omega = 0$.
3. It should enable integration by parts, which can be achieved with the help of the Leibniz rule and the relation

$$\int \mathrm{d}^D x\,{}^\star\Omega\,{}^\star D_{\pm\mu} V^\mu = \int \mathrm{d}^D x \partial_\mu({}^\star\Omega V^\mu)\,, \tag{42}$$

so that we are able to use Stoke's theorem.

For Riemann and Riemann–Cartan space-times, the solution for the measure factor is well known: $\Omega = \sqrt{-G}$

($G = \det G_{\mu\nu}$). For spaces with non-metricity, this standard measure is not preserved under parallel transport, and requirements 2. and 3. are not satisfied. Instead of changing the connection and finding the volume-preserving one, as has been done [1, 2], we prefer to change the measure.

Let us try to find the stringy measure in the form ${}^\star\Omega = \Lambda(x)\sqrt{-G}$. In order to be preserved under parallel transport with the stringy connection, it must satisfy the condition

$$ {}^\star D_{\pm\mu}(\sqrt{-G}\Lambda) = \partial_\mu(\sqrt{-G}\Lambda) - {}^\star\Gamma^\rho_{\pm\mu\rho}\sqrt{-G}\Lambda = 0\,. \tag{43} $$

Using the relation

$$ {}^\star\Gamma^\rho_{\pm\mu\rho} = \partial_\mu \ln\left(\sqrt{-G}\,\mathrm{e}^{-2\varphi}\right) = \Gamma^\rho_{\pm\mu\rho} + \frac{D}{2}{}^\star q_\mu\,, \tag{44} $$

we find the equation for Λ: $\partial_\mu\Lambda = \frac{D}{2}{}^\star q_\mu\Lambda$. The fact that the stringy Weyl vector ${}^\star q_\mu$ is the gradient of the scalar field φ, defined in (30), helps us to find the solution $\Lambda = \mathrm{e}^{-2\varphi}$. The stringy measure factor, preserved under parallel transport with the connection ${}^\star\Gamma^\mu_{\pm\nu}$, obtains the form

$$ {}^\star\Omega = \sqrt{-G}\,\mathrm{e}^{-2\varphi}\,. \tag{45} $$

Consequently, we have ${}^\star\Gamma^\rho_{\pm\mu\rho} = \partial_\mu \ln {}^\star\Omega$, and (42) is satisfied. So, if we use the stringy measure ${}^\star\Omega$, we can integrate by parts, and all requirements are satisfied.

The above measure is a volume-form compatible with the connection of [17]. In our case only non-metricity contributes to the improvement, because in stringy geometry the torsion contribution vanishes, ${}^\star T_\pm{}^\rho{}_{\mu\rho} = 0$.

The measure factor in (40), $\sqrt{-G}\,\mathrm{e}^{-2\Phi}$, has the same form as the one in the present paper and confirms the existence of some space-time non-metricity. The requirement of the full measure equality, $\varphi = \Phi$, leads to a Liouville-like equation for the dilaton field:

$$ G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \mathrm{e}^{-4\Phi} = 0\,. \tag{46} $$

For $D = 2$ it turns out to be the actual Liouville equation.

4.2 The space-time Lagrangian

We are going to reproduce the Lagrangian defined in (40) with suitable combinations of the stringy scalar curvature, defined in the standard way with the stringy connection (12),

$$ \begin{aligned} {}^\star R_\pm &= R - B^2 + 2D^2\varphi - 4(\partial\varphi)^2 \\ &\quad + \mathrm{e}^{4\varphi}\big[2(aB)^2 + 2a^\mu\partial_\mu(Da) \\ &\quad + a^\mu D_\mu(Da) + (Da)^2\big]\,, \end{aligned} \tag{47} $$

the stringy torsion (27),

$$ {}^\star T^\rho{}_{\pm\mu\nu} = \pm\big[2B^\rho{}_{\mu\nu} - 2\,\mathrm{e}^{4\varphi}a^\rho(aB)_{\mu\nu}\big]\,, \tag{48} $$

and the stringy non-metricity (28),

$$ {}^\star Q_{\pm\mu\rho\sigma} = \mathrm{e}^{4\varphi}[D_\mu(a_\rho a_\sigma) \mp a_\rho(aB)_{\sigma\mu} \mp a_\sigma(aB)_{\rho\mu}]\,, \tag{49} $$

where $(aB)_{\mu\nu} = a^\rho B_{\rho\mu\nu}$, $(aB)^2 = a^\rho B_{\rho\mu\nu}a^\sigma B_\sigma{}^{\mu\nu}$ and $Da = D_\mu a^\mu$. First, we construct the corresponding invariants

$$ \begin{aligned} {}^\star T^2_\pm &\equiv {}^\star T_{\pm\mu\nu\rho}{}^\star T^{\mu\nu\rho}_\pm \\ &= 4[B^2 - \mathrm{e}^{4\varphi}(aB)^2]\,, \end{aligned} \tag{50} $$

$$ \begin{aligned} {}^\star Q^2_\pm &\equiv {}^\star Q_{\pm\mu\nu\rho}{}^\star Q^{\mu\nu\rho}_\pm \\ &= 8(\partial\varphi)^2 + 2\,\mathrm{e}^{4\varphi}\left[(D_\mu a_\nu)(D^\mu a^\nu) + (aB)^2\right] \end{aligned} \tag{51} $$

and

$$ \begin{aligned} {}^\star q^2 &\equiv {}^\star q_\mu{}^\star q^\mu \\ &= \frac{1}{D^2}G^{\rho\sigma}Q_{\pm\mu\rho\sigma}G^{\varepsilon\eta}Q_\pm{}^\mu{}_{\varepsilon\eta} = \frac{16}{D^2}(\partial\varphi)^2\,, \end{aligned} \tag{52} $$

where ${}^\star q_\mu$ is stringy Weyl vector defined in (29). Note that all invariants are independent on the $\pm$ indices, and we put ${}^\star R_\pm = {}^\star R$, ${}^\star T^2_\pm = {}^\star T^2$ and ${}^\star Q^2_\pm = {}^\star Q^2$.

We assume that the Lagrangian is linear in these invariants and choose appropriate coefficients in front of them,

$$ {}^\star\mathcal{L} \equiv {}^\star R + \frac{1}{48}\left(11\,{}^\star T^2 - 26\,{}^\star Q^2\right) + \frac{1}{3}\left(\frac{5D}{4}\right){}^\star q^2\,, \tag{53} $$

in order to reproduce the expression (40):

$$ \begin{aligned} {}^\star\mathcal{L} = R &- \frac{1}{12}B^2 + 2D^2\varphi + \frac{1}{a^2}\Big[2a^\mu\partial_\mu(Da) + a^\mu D_\mu(Da) \\ &+ (Da)^2 - \frac{13}{12}(D_\mu a_\nu)(D^\mu a^\nu)\Big]\,. \end{aligned} \tag{54} $$

If the condition (46) is satisfied, the Lagrangian (54), up to the term with the factor $\frac{1}{a^2}$, coincides with that defined in (40).

The Lagrangian (40) has been obtained from one-loop perturbative computations. The higher-loop corrections generally depend on the renormalization scheme [22]. We argue that the term proportional to $\frac{1}{a^2}$ in ${}^\star\mathcal{L}$ originates from higher order contributions. The reason is that there is a difference between the Lagrangian and Hamiltonian perturbative approaches [16]. The leading order term of the Hamiltonian contains a Φ-dependent part proportional to $\frac{1}{a^2}$, while the leading order term of the Lagrangian is Φ-independent. Because the stringy invariants of the present paper are defined by the Hamiltonian form of the field equations, (9)–(11), we expect that the term proportional to $\frac{1}{a^2}$ is a consequence of different perturbative approaches. Up to this term, for $\varphi = \Phi$ we have ${}^\star\mathcal{L} = \mathcal{L}$.

5 Conclusions

In the present paper we show that the probe string, as an extended object, can see more space-time features than the probe particle – torsion and non-metricity. We find their forms in terms of the background fields, which define the target space geometry felt by the string.

The equations of motion (9)–(11) help us to obtain the explicit expression for the stringy connection (12). It produces the stringy torsion (27) and the non-metricity (28), originating from the antisymmetric field $B_{\mu\nu}$ and the dilaton fields Φ, respectively.

Let us clarify how the space-time geometry depends on the background fields. In the presence of the metric tensor $G_{\mu\nu}$, the space-time is of Riemann type. Inclusion of the antisymmetric field $B_{\mu\nu}$ produces a Riemann–Cartan space-time. The appearance of the dilaton field Φ breaks the compatibility between the metric tensor and the stringy connection. When all three background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ are present, the string feels the complete stringy space-time.

Finally, we construct the bosonic string space-time action in terms of geometrical quantities. In order to find the integration measure that is invariant under parallel transport, we used the fact that the stringy Weyl vector is the gradient of the scalar field φ. We also derive the Lagrangian as a function of the stringy invariants: scalar curvature, torsion and non-metricity.

We discuss the connection between our result and that of [3–7], in spite of their different origins. The standard result is quantum and perturbative, while our's is classical and non-perturbative. In particular, our scalar field φ, defined in (30), plays the role of a dilaton field Φ and has the same position in all expressions. Up to the non-linear term proportional to $\frac{1}{a^2}$ (which is a consequence of the different perturbation theories in the Lagrangian and Hamiltonian approaches) for $\varphi = \Phi$, these two actions are equal, including the dilaton factor in the integration measure.

It is well known that the dilaton dependent Weyl transformation

$$G^{\mathrm{E}}_{\mu\nu} = \mathrm{e}^{-\frac{2(\Phi_0 - \Phi)}{D-2}} G_{\mu\nu} \tag{55}$$

takes the Lagrangian to the Hilbert form:

$$S^{\mathrm{E}} = \int \mathrm{d}x \sqrt{-G^{\mathrm{E}}} \left[R - \frac{1}{12} \mathrm{e}^{-\frac{8\Phi}{D-2}} B^2 - \frac{4}{D-2} (\partial\Phi)^2 \right]_{\mathrm{E}} , \tag{56}$$

where the index E means that all quantities are define in terms of the Einstein metric, $G^{\mathrm{E}}_{\mu\nu}$. In this form of the Lagrangian, the dilaton decouples from the curvature, but it is still coupled to the torsion through the second term. As a consequence, neither of the two Lagrangians obeys the equivalence principle, so that the change from the string frame to the Einstein one does not help us to choose a preferred definition of the metric [19].

Our approach prefers the so called string frame to be taken as more fundamental, because we can offer a clear geometrical interpretation for it. In particular, the preservation of the integration measure under parallel transport singles out the form (45) for it. This is just characteristic of the string frame.

There is another reason for the string frame [18, 19]. Only when the action is written in terms of the fields originating from strings, the constant part of the dilaton, Φ_0, appears as an overall factor as well as the coupling constant in Yang–Mills theories.

Consequently, we show that the string space-time action, in terms of geometrical quantities, depends not only on the curvature and torsion felt by the probe string, but also on the non-metricity, which causes absence of the equivalence principle.

Let us mention one curiosity. It is known that the coefficient in front of the Liouville action is proportional to the central charge and measures quantum breaking of the classical symmetry. The contributions to the central charge of the anticommuting ghosts b, c corresponding to the conformal symmetry and the commuting ghosts β, γ corresponding to the superconformal symmetry are $\frac{-26}{48}$ and $\frac{11}{48}$, respectively. In the definition of the Lagrangian ${}^\star\mathcal{L}$, (53), the coefficients in front of the stringy non-metricity and the stringy torsion are just equal to the coefficients of the b, c and β, γ ghost contributions. We do not find a good reason for this similarity, but we find it interesting to mention this coincidence.

Appendix: World-sheet geometry

We use the notation of [16], expressing the intrinsic world-sheet metric tensor $g_{\alpha\beta}$ in terms of the light-cone variables (h^+, h^-, F):

$$g_{\alpha\beta} = \mathrm{e}^{2F} \hat{g}_{\alpha\beta} = \frac{1}{2} \mathrm{e}^{2F} \begin{pmatrix} -2h^- h^+ & h^- + h^+ \\ h^- + h^+ & -2 \end{pmatrix} . \tag{A.1}$$

The world-sheet interval has the form

$$\mathrm{d}s^2 = g_{\alpha\beta} \, \mathrm{d}\xi^\alpha \, \mathrm{d}\xi^\beta = 2 \, \mathrm{d}\xi^+ \, \mathrm{d}\xi^- , \tag{A.2}$$

where

$$\mathrm{d}\xi^\pm = \frac{\pm 1}{\sqrt{2}} \mathrm{e}^F (\mathrm{d}\xi^1 - h^\pm \, \mathrm{d}\xi^0) = e^\pm{}_\alpha \, \mathrm{d}\xi^\alpha . \tag{A.3}$$

The quantities $e^\pm{}_\alpha$ define the light-cone one-form basis, $\theta^\pm = e^\pm{}_\alpha \, \mathrm{d}\xi^\alpha$, and its inverse defines the tangent vector basis, $e_\pm = e_\pm{}^\alpha \partial_\alpha = \partial_\pm$.

In the tangent basis notation, the components of the arbitrary vector V_α have the form

$$V_\pm = e_\pm{}^\alpha V_\alpha = \frac{\sqrt{2} \mathrm{e}^{-F}}{h^- - h^+} (V_0 + h^\mp V_1) . \tag{A.4}$$

The world-sheet covariant derivatives of the tensor X_n are

$$\nabla_\pm X_n = (\partial_\pm + n\omega_\pm) X_n , \tag{A.5}$$

where the number n is the sum of the indices, counting the index + with 1 and the index − with −1. The two dimensional covariant derivative $\nabla_\pm$ is defined with respect to the connection

$$\omega_\pm = \mathrm{e}^{-F} (\hat{\omega}_\pm \mp \hat{\partial}_\pm F) , \qquad \hat{\omega}_\pm = \mp \frac{\sqrt{2}}{h^- - h^+} h^{\mp\prime} . \tag{A.6}$$

References

1. M. Blagojević, Gravitation and gauge symmetries (IoP Publishing, Bristol, 2002)
2. F.W. Hehl, J.D. McCrea, E.W. Mielke, Y. Ne'eman, Phys. Rep. **258**, 1 (1995)
3. C.G. Callan, D. Friedan, E.J. Martinec, M.J. Perry, Nucl. Phys. B **262**, 593 (1985)
4. E.S. Fradkin, A.A. Tseytlin, Phys. Lett. B **158**, 316 (1985)
5. E.S. Fradkin, A.A. Tseytlin, Nucl. Phys. B **261**, 1 (1985)
6. T. Banks, D. Nemeschansky, A. Sen, Nucl. Phys. B **277**, 67 (1986)
7. C.G. Callan, I.R. Klebanov, M.J. Perry, Nucl. Phys. B **278**, 78 (1986)
8. T. Dereli, R.W. Tucker, Class. Quantum Grav. **4**, 791 (1987)
9. T. Dereli, R.W. Tucker, Class. Quantum Grav. **11**, 2575 (1994)
10. T. Dereli, R.W. Tucker, Class. Quantum Grav. **12**, L31 (1995)
11. H. Cebeci, T. Dereli, Phys. Rev. D **71**, 024 016 (2005)
12. A. Saa, Class. Quantum Grav. **12**, L85 (1995)
13. J. Scherk, J.H. Schwarz, Phys. Lett. B **52**, 347 (1974)
14. J. Scherk, J.H. Schwarz, Nucl. Phys. B **81**, 118 (1974)
15. E. Braaten, T.L. Curtright, C.K. Zachos, Nucl. Phys. B **260**, 630 (1985)
16. B. Sazdović, IJMP A **20**, 5501 (2005) [hep-th/0 304 086]
17. A. Saa, J. Geom. Phys. **15**, 102 (1995)
18. M.B. Green, J.H. Schwarz, E. Witten, Superstring Theory (Cambridge University Press, Cambridge, 1987)
19. J. Polchinski, String Theory (Cambridge University Press, Cambridge, 1998)
20. T.L. Curtright, C.K. Zachos, Phys. Rev. Lett. **53**, 1799 (1984)
21. S. Mukhi, Phys. Lett. B **162**, 345 (1985)
22. A.A. Tseytlin, Int. J. Mod. Phys. A **4**, 1257 (1989)